Misconceptions Reconceived: A Constructivist Analysis of Knowledge in Transition

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Misconceptions Reconceived: A Constructivist Analysis of Knowledge in Transition

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This article uses a critical evaluation of research on student misconceptions in science and mathematics to articulate a constructivist view of learning in which student conceptions play productive roles in the acquisition of expertise. We acknowledge and build on the empirical results of misconceptions research but question accompanying views of the character, origins, and growth of students' conceptions. Students have often been viewed as holding flawed ideas that are strongly held, that interfere with learning, and that instruction must confront and replace. We argue that this view overemphasizes the discontinuity between students and expert scientists and mathematicians, making the acquisition of expertise difficult to conceptualize. It also conflicts with the basic premise of constructivism: that students build more advanced knowledge from prior under-standings. Using case analyses, we dispute some commonly cited dimensions of discontinuity and identify important continuities that were previously ignored or underemphasized. We highlight elements of knowledge that serve both novices and experts, albeit in different contexts and under different conditions. We provide an initial sketch of a constructivist theory of learning that interprets

students' prior conceptions as resources for cognitive growth within a complex systems view of knowledge. This theoretical perspective aims to characterize the interrelationships among diverse knowledge elements rather than identify particular flawed conceptions; it emphasizes knowledge refinement and reorganization, rather than replacement, as primary metaphors for learning; and it provides a framework for understanding misconceptions as both flawed and productive.

The idea that students develop misconceptions lies at the heart of much of the empirical research on learning mathematics and science of the last 15 years. Following Piaget's repeated demonstrations that children think about the world in very different ways than do adults, educational researchers in the late 1970s began to listen carefully to what students were saying and doing on a variety of subject-matter tasks. What they heard and subsequently reported was both surprising and disturbing: Students had ideas that competed, often quite effectively, with the concepts presented in the classroom. Students did not come to instruction as blank slates. They had developed durable conceptions with explanatory power, but those conceptions were inconsistent with the accepted mathematical and scientific concepts presented in instruction.

In this article, we examine the misconceptions view of mathematical and scientific thinking and learning represented in nearly 2 decades of empirical research. This perspective has significantly advanced our understanding of learning by producing detailed characterizations of the understandings students bring to instruction, thereby highlighting the deep and complex changes involved in acquiring expertise. However, fundamental problems arise when most students' ideas are characterized as misconceptions. Misconceptions have generally been seen as mistakes that impede learning, a view that is difficult to square with the premise that students construct their mathematical and scientific knowledge. Further advances in understanding learning will depend on rethinking the role that students' conceptions of mathematical and scientific phenomena play in sophisticated, expert-like knowledge. Where misconceptions research has focused on the discontinuities between novice students and experts, we identify and emphasize important dimensions of continuity between them.

The basic stance that underlies our reinterpretation is constructivism—the view that all learning involves the interpretation of phenomena, situations, and events, including classroom instruction, through the perspective of the learner's existing knowledge. This epistemology serves both a critical role in evaluating the main themes of misconceptions research and a constructive role as the foundation for our own theoretical proposal. Most current accounts of learning as a process of individual construction provide a general orientation for, but fall well short of, a theory of learning mathematical and scientific concepts of modest complexity (Resnick, 1987; Smith, 1990; von

Glasersfeld, 1987). They do not provide sufficiently detailed accounts of how students' continual reconstruction of their existing knowledge may produce the observed intermediate states of understanding and eventual mastery of a domain. The constructivist principles that we outline clarify the role of misconceptions in learning and extend constructivism beyond its basic epistemological premise. They conceptualize knowledge not in terms of the presence or absence of single elements (e.g., $F = ma$ or conversion to common denominator) but as knowledge systems composed of many interrelated elements that can change in complex ways. This knowledge system framework makes it easier to understand how novice conceptions can play productive roles in evolving expertise, despite their flaws and limitations.

THE IMPACT OF MISCONCEPTIONS RESEARCH

Two major motivations underlie our reevaluation of misconceptions research. The first is to acknowledge the valuable insights about subject-matter learning contributed by studies of students' misconceptions, which have focused attention on what students actually say and do in a wide variety of mathematical and scientific domains. This work represents a fundamental advance from previous approaches that essentially divided student responses into two categories, correct and incorrect (e.g., Bloom, 1976; Gagne, 1968). Those dichotomous evaluations of student responses hid from view the systematicity and underlying conceptual sense of student errors. Building on Piaget's studies of cognitive development, misconceptions research freed investigators' descriptive capabilities and legitimized their efforts to uncover structure and meaning in students' responses. These investigations have produced careful characterizations of students' conceptions in a variety of conceptual domains and their changes (or not) in response to experience. Although we recognize these fundamental advances, we think it is time to move beyond simple models of knowledge and learning in which novice misconceptions are replaced by appropriate expert concepts.

The other major motivation is the impressive impact that 15 years of misconceptions research has had on educational research and practice in mathematics and science. From a handful of investigations in a small number of science domains in the mid-1970s, research expanded to nearly every domain of science and to mathematics and computer programming as well by the mid-1980s. The corpus of empirical results has grown to the point that review articles of substantial length are required to properly survey the field (Confrey, 1990; Eylon & Linn, 1988). The central theoretical term *misconception* is widely used to describe and explain students' performance in specific subject-matter domains (Eaton, Anderson, & Smith, 1983; Gardner, 1991; Shaughnessy, 1992). Research on domain-specific learning in both classroom and laboratory settings is now designed and conducted under the assumption that students' misconceptions in these domains must be taken seriously (e.g., Nesher, 1987).

Recent work in assessment also reflects a growing sensitivity to the importance of student misconceptions. In mathematics, for example, the analyses of recent National Assessment of Educational Progress results (Lindquist, 1989) and preliminary results from open-ended California Assessment Program (CAP) test items (California State Department of Education, 1989) both show that misconceptions have a strong influence on how student learning is currently evaluated. Whereas only 15 years ago researchers simply separated correct responses and errors, it is now common, even in large-scale assessments, to actively search for misconceptions to explain frequent student errors. In recognition of the substantial impact of misconceptions on educational research and practice, we examine the main themes of misconceptions research.

THE CENTRAL ASSERTIONS OF
MISCONCEPTIONS RESEARCH

Because we accept the empirical results of misconceptions research, the most sensible way to critically assess this research tradition would be to examine its theoretical commitments. But most misconceptions research has focused more on the description of students' ideas—before, during, and after instruction—and much less on the development of theoretical frameworks that relate those ideas to the process of learning.[1] It is possible, however, to identify general assertions about learning that are commonly stated in the misconceptions literature. We identify seven such assertions, which we state with supporting examples and references.

Identifying these central assertions requires judgment, and it might be objected that we have falsely attributed a common theory of misconceptions to a diverse group of researchers. We recognize the substantial theoretical diversity among the investigators whose work we cite as misconceptions research. Some researchers who have stated one or more of the central assertions will certainly doubt or disagree with others. Our purpose is not to evaluate the theoretical perspectives of individual researchers but to examine a set of assertions that is frequently expressed in the corpus of misconceptions literature. Those assertions collectively represent a coherent theoretical position that merits our critical attention.

Students Have Misconceptions

This assertion rejects the *tabula rasa* view of students before instruction. Before they are taught expert concepts, students have conceptions that explain some of the mathematical and scientific phenomena that expert

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[1]There are some well-known exceptions to this generalization. For example, the work of Strike, Posner, and colleagues (e.g., Posner, Strike, Hewson, & Hertzog, 1982; Strike & Posner, 1985) has focused precisely on providing a general model of conceptual change that can account for the evolution of students' understanding of science from misconceptions toward expertise.

concepts explain, but these conceptions are *different* from the currently accepted disciplinary concepts presented in instruction. Because they regularly differ from instructed concepts and guide students' reasoning, education in mathematics and science must take them seriously.

The observed differences between student ideas and corresponding expert concepts created the need for some theoretical terminology to characterize those differences. Misconceptions research, in fact, generated a wide variety of terms to characterize students' conceptions, including *preconceptions* (Clement, 1982b; Glaser & Bassok, 1989; Wiser, 1989), *alternative conceptions* (Hewson & Hewson, 1984), *naive beliefs* (McCloskey, Caramazza, & Green, 1980), *alternative beliefs* (Wiser, 1989), *alternative frameworks* (Driver, 1983; Driver & Easley, 1978), and *naive theories* (McCloskey, 1983; Resnick, 1983), as well as the standard term misconception. Though these terms have all asserted fundamental differences between students and experts, the variation among them reflects differences in how researchers have characterized the cognitive properties of student ideas and their relation to expert concepts.[2] In this article, we use the most common term—misconception—to designate a student conception that produces a systematic pattern of errors.[3] Instruction in mathematics and science poses problems for students to solve and phenomena for them to explain. *Conceptions* (or *ideas*) identify and relate factors that students use to explain intriguing or problematic phenomena. They also represent the knowledge, expressed in terms of solution strategies and their rationale, that constitutes the core solution to specific problems.

Misconceptions Originate in Prior Learning

Misconceptions arise from students' prior learning, either in the classroom (especially for mathematics) or from their interaction with the physical and social world. In Newtonian mechanics—perhaps the domain most extensively analyzed—researchers have agreed that students' misconceptions

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[2]The qualifiers, *pre, mis,* and *alternative,* each point to different presumptions about the nature and origin of conceptions. Likewise, *belief* and *conception* suggest unitary cognitive structures, whereas *theory* and *framework* embed conceptions in larger scale structures. Expanding slightly from the misconceptions perspective, we note that distinctions between *formal* and *informal* knowledge (e.g., Hiebert & Behr, 1988) and *expert* and *novice* (e.g., Glaser & Chi, 1988) also have been used to characterize and emphasize fundamental distinctions between student conceptions and expert concepts. In Appendix A, we present a more extensive discussion of how these terms arise from different epistemological orientations and frame the study of learning in different ways.

[3]Researchers' ability to infer explicitly the content of such hypothesized student conceptions from observed patterns of student errors is central to this formulation. Some uses of misconception in the educational literature simply designate a pattern of errors and do not completely satisfy this definition (e.g., California State Department of Education, 1989).

about force and motion are the result of day-to-day experiences in the physical world (Clement, 1983, 1987; McCloskey, 1983; Resnick, 1983). As Clement (1983) explains, the persistence of the "motion implies a force" misconception "is rooted in everyday perceptual-motor experiences with pushing and pulling objects" (p. 337). In elementary mathematics, misconceptions usually originate in prior instruction as students incorrectly generalize prior knowledge to grapple with new tasks (Nesher, 1987; Resnick et al., 1989).[4] For example, in their efforts to understand the ordering of decimal fractions with *ragged* fractional parts (e.g., 3.4 and 3.671), many middle-school students apply prior knowledge of either whole numbers or common fractions (Nesher & Peled, 1986; Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985). Students who apply whole-number knowledge to compare 3.4 and 3.671 ignore the decimal points and treat each as whole numbers, concluding that 3.671 is greater than 3.4. Other students use their knowledge of fractions, focusing on the size of the smallest decimal place in the numbers but ignoring the relative value of the digits in those locations. Because 3.671 has one thousandth and thousandths are smaller than tenths, they concluded that 3.671 is greater than 3.4.

Misconceptions Can Be Stable and Widespread Among Students. Misconceptions Can Be Strongly Held and Resistant to Change.

Rather than being momentary conjectures that are quickly discarded, misconceptions consistently appear before and after instruction in substantial numbers of students and adults in a wide variety of subject-matter domains and are often actively defended. As Clement has shown for Newtonian mechanics and elementary algebra, the same mistaken reasoning can appear in a variety of different problem contexts. "Motion implies a force" is held responsible for errors in student reasoning on problems involving swinging pendulums, coin flipping, and orbiting rockets (Clement, 1983). Similarly, college students make the same reversal error in translating multiplicative relationships into equations (e.g., translating "there are four people ordering cheesecake for every five people ordering strudel" into "4C = 5S"), whether the initial relations were stated in sentences, pictures, or data tables (Clement, 1982a). In domains of multiplication (Fischbein, Deri, Nello, & Marino, 1985), probability (Shaughnessy, 1977), and algebra (Clement, 1982a; Rosnick, 1981), mis-

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[4]We do not mean to suggest that the origins of mathematical misconceptions are always prior classroom instruction and that the origins of scientific misconceptions are always everyday physical experiences. There are, for example, always everyday, out-of-school experiences that are relevant to understanding new mathematical ideas.

conceptions continue to appear even after the correct approach has been taught. Sometimes misconceptions even coexist alongside the correct approach (Clement, 1982a).

Misconceptions also appear widely in the student and adult population. Wiser (1989) reported that middle-school students' flawed source–recipient model of kinetic phenomena shows "a high degree of consistency within and between students" (p. 11). Cohen, Eylon, and Ganiel (1983) found that large numbers of their high school students and physics teachers harbored misconceptions about simple circuits such as "a battery delivers a constant current," regardless of the circuit's components. Tversky and Kahneman (1982) demonstrated that misconceptions about statistics and probability such as the *representativeness heuristic* and the *law of small numbers,* are commonly asserted, even by professionals.

Perhaps most troubling is that students can doggedly hold onto mistaken ideas even after receiving instruction designed to dislodge them. Summarizing years of misconceptions research in Newtonian mechanics, Clement (1987) reported, *"Many preconceptions are deep seated and resistant to change"* (p. 3). Similarly, the naive theories of motion McCloskey (1983) ascribed to his students were "consistent across individuals," "very strongly held," and "not easily changed by classroom instruction." This persistence does not necessarily mean that instruction has failed completely. It can succeed in imparting the correct concept that then *competes with* the prior misconception, as Fischbein et al. (1985) note in their analysis of students' notions of multiplication.

> The initial didactical models [the misconceptions from previous instruction] seem to become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct. (p. 16)

Misconceptions Interfere With Learning

Because of their strength and flawed content, misconceptions interfere with learning expert concepts. Hiebert and Behr (1988) interpreted a number of studies as showing that middle school students' numerical knowledge of additive relations has interfered with learning various multiplicative relations such as proportional reasoning. The source of students' difficulty in learning formal notions of probability has been attributed to the interference of misconceptions about probabilistic events such as the law of small numbers (Lindquist, 1989; Shaughnessy, 1985). For Cohen et al. (1983), misconceptions about circuits impede the rigorous study of electricity. Researchers in physics have reported that misconceptions even cause students to misperceive laboratory events and classroom demonstrations (Clement, 1982b; Resnick, 1983).

Misconceptions Must Be Replaced

Because misconceptions are so prevalent, learning mathematics and science must involve a shift away from misconceptions to expert concepts. This shift is often characterized as replacement: More adequate expert ideas must be developed and replace existing misconceptions. Learning involves both the acquisition of expert concepts and the dispelling of misconceptions. The assumption that removing misconceptions has no negative consequences because they play no productive role in expertise is implicit in the replacement view.

Multiple lines of evidence show that the replacement view of conceptual change is frequently expressed in misconceptions literature: (a) Explicit statements of replacement are not uncommon, (b) explicit rejections of replacement are difficult to locate, (c) alternative models of change are either absent or undeveloped, and (d) replacement is consistent with other assertions about misconceptions (discussed later). These lines of argument and the evidence in support of them are developed in Appendix B. The evidence presented there shows that this assertion, despite its bold simplicity, is not simply a rhetorical straw man.

Instruction Should Confront Misconceptions

To neutralize the interference of misconceptions, instruction should confront students with the disparity between their misconceptions and expert concepts. When the disparity becomes explicit, students will appreciate the advantages of the expert concepts and give up their misconceptions. Researchers who developed classroom approaches to misconceptions have often proposed rational competition between misconceptions and corresponding expert concepts. This instruction first has students articulate their unconscious misconceptions and then establishes a framework for comparing the validity of the competing ideas (Champagne, Gunstone, & Klopfer, 1985; Strike & Posner, 1985). Confrontation begins as an external, social interaction in the classroom, but for confrontation to succeed, the competition between misconceptions and expert concepts must be internalized by students. Confrontation and replacement are therefore inextricably linked: Successful instructional confrontation leads to learning by replacement.

The related metaphor of *overcoming* has also been used to describe the process of conceptual change (Brown & Clement, 1989; Shaughnessy, 1982). Although overcoming suggests a similar competition between ideas, researchers who advocate it suggest how to leverage the competition so that students will be more likely to give up their misconceptions. Brown and Clement's *bridging analogies* establish and strengthen connections between expert physics concepts and other existing student conceptions, thus undermining the competing misconceptions. Like interference and replacement,

overcoming suggests that misconceptions are strictly a hindrance to expert reasoning and therefore should be discarded.[5]

Research Should Identify Misconceptions

A major task for research in mathematics and science learning is to document misconceptions in as many subject-matter domains as possible. Though we have not found explicit statements of this assertion in the literature, we have inferred it from the character of research that appeared after early reports of misconceptions attracted researchers' attention. The focus of the later work was to identify misconceptions in yet another domain of science or mathematics. Much less emphasis was given to modeling the learning of successful students in those domains, to characterizing how misconceptions (and the cognitive structures that embed them) evolve, or to describing the nature of instruction that successfully promotes such learning. When we consider the corpus of misconceptions research, the major research effort has been to identify more misconceptions.

CONFLICTS WITH CONSTRUCTIVISM

Although constructivist principles have not been explicit in these central assertions, some researchers who have analyzed misconceptions have also advocated this general perspective on learning (Driver, 1983; Nesher, 1987; Resnick, 1987). Constructivism emphasizes the role of prior knowledge in learning. Students interpret tasks and instructional activities involving new concepts in terms of their prior knowledge. Errors are characteristic of initial phases of learning because students' existing knowledge is inadequate and supports only partial understandings. As their existing knowledge is recognized to be inadequate to explain phenomena and solve problems, students learn by transforming and refining that prior knowledge into more sophisticated forms. Substantial conceptual change does not take place rapidly, and relatively stable intermediate states of understanding often precede conceptual mastery.

Our central claim is that many of the assertions of misconceptions research are inconsistent with constructivism. Misconceptions research has emphasized the flawed results of student learning. Constructivism, in contrast, characterizes the process of learning as the gradual recrafting of existing knowledge that, despite many intermediate difficulties, is eventually successful. It is difficult to see how misconceptions that (a) interfere with learning, (b) must be replaced, and (c) resist instruction can also play the role

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[5]Brown and Clement are exceptional among misconceptions researchers in attributing an explicit productive role, as the anchors of analogies, to some prior student ideas. Minstrell (1989) and some of our own prior work (diSessa, 1983) have also illustrated the productive roles played by student ideas about the behavior of the physical world.

of useful prior knowledge that supports students' learning. If we take constructivism seriously, we must either reconsider the solely mistaken character of misconceptions or look for other ideas to serve as productive resources for student learning. We now attempt to sharpen this argument by pointing to specific parts of the misconceptions perspective that conflict with constructivism.

Deepening the Learning Paradox

Some cognitive researchers have suggested that our common-sense notions of learning are not logically consistent (discussion in Bereiter, 1985; Fodor, 1980). How is it possible for our existing cognitive structures to transform themselves into more complex forms? Fodor's answer is that it is not possible. More powerful cognitive structures can be learned only if they existed in some form already. What appear to educators as the results of learning are simply the emergence of cognitive structures that existed in nascent form from birth.

The emphasis in the misconceptions perspective on the differences between students and experts significantly strengthens this paradox by widening the gulf that a constructivist theory of learning must bridge. In focusing only on how student ideas conflict with expert concepts, the misconceptions perspective offers no account of productive ideas that might serve as resources for learning.[6] Because they are fundamentally flawed, misconceptions themselves must be replaced. What additional relevant ideas students might have available remains a mystery. An account of useful resources that are marshaled by learners is an essential component of a constructivist theory, but the misconceptions perspective fails to provide one.

Piaget's theory of cognitive development attempted to resolve this problem without appealing to preformed and content-specific cognitive structures (Piaget, 1971). Like Piaget, we accept that a major task for a constructivist theory of learning is to present a psychologically plausible resolution of the learning paradox. Like Bereiter (1985), we suggest that a convincing response depends on identifying a range of cognitive resources that can support the bootstrapping of more advanced cognitive structures. Our use of the term *resources* designates any feature of the learner's present cognitive state that can serve as significant input to the process of conceptual growth. Because our focus is learning mathematics and science, the resources we emphasize are students' existing understandings of their mathematical and physical worlds.

We suggest that misconceptions, especially those that are most robust, have their roots in productive and effective knowledge. The key is context—

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[6]Again, there are exceptions to this deficit (e.g., Clement, Brown, & Zietsman's 1989 analysis of anchoring conceptions in learning Newtonian mechanics), but such exceptions are rare.

where and how those conceptions are used. There are certainly contexts in which students' existing knowledge is ineffective, or more carefully articulated mathematical and scientific knowledge is unnecessary. Moreover, some flawed conceptions, though sensible in the short run, may play no role whatsoever in more expert reasoning. But the major limitation of misconceptions research has been its examination of a restricted set of contexts in which students' conceptions fail while leaving unidentified a broader range of contexts in which they are productive.

Is Replacement an Adequate Model of Learning?

Much misconceptions research has suggested that learning is a process of replacing misconceptions with appropriate expert knowledge. Often replacement is characterized as a one-for-one process. The *motion implies a force* misconception should be replaced by Newton's Second Law, $F = ma$, or following Petrie (1976), the impetus theory should be replaced by Newton's First Law. There are two main reasons to doubt replacement, either as a cognitive process or as a useful metaphor for learning: empirical evidence of the complexity of knowledge and learning and considerations of theoretical consistency.

The plausibility of replacement depends on very simple models of knowledge. Misconceptions (and expert conceptions) are taken to be unitary, independent, and therefore separable cognitive elements. Learning is a process of removing misconceptions from students' cognitive structures and inserting appropriate expert concepts in their place. However, the relationship between particular conceptions and the cognitive structures that embed them are far more complicated than such unitary models suggest (Clement, 1982a, 1987; Kuhn & Phelps, 1982; Schoenfeld, Smith, & Arcavi, 1993). Clement's studies in mechanics and multiplicative equations have shown that students can shift between correct and flawed approaches within the same problem-solving episode, which suggests that cognitive structures can embrace both expert concepts and misconceptions. If concepts are more like complex clusters of related ideas than separable independent units, then replacement looks less plausible as a learning process (diSessa, in press; Smith, 1992).

Replacement also conflicts with the constructivist premise that learning is the process of adapting prior knowledge. The critical question raised by replacement is: What prior knowledge is involved in the construction of the expert concepts that replace misconceptions? If we accept the mistaken character of misconceptions, they cannot serve as resources. The other possibility is that students have some complementary pool of productive knowledge that can be brought into competition with misconceptions, but misconceptions researchers have not identified such resources within the novice understanding.

Is Confrontation an Appropriate Model of Classroom Learning?

For classroom instruction to be successful in confronting misconceptions, teachers must present expert concepts in clear opposition to students' faulty conceptions. This instruction must include demonstrations and activities that produce counterevidence and plausible conceptual alternatives to target misconceptions. The confrontation of ideas in the classroom is then internalized by students as a psychological process of competition that finally results in the replacement of the misconception.

There are both strengths and weaknesses in this conceptualization of classroom instruction. We need energetic classroom discussions in which students take positions, make sense of and explain problematic phenomena, and articulate justifications for their ideas. Activities that produce states of cognitive conflict are certainly desirable and conducive to conceptual change. However, as judged by constructivist standards, confrontation suffers from important deficits as either a phase of conceptual change or a model of instruction. As cognitive competition, it cannot explain why expert ideas win out over misconceptions. The rational replacement of one conception with another requires criteria for judgment. As knowledge, those criteria must be constructed by the learner, and neither confrontation nor replacement explains the origins of such principles for choosing concepts, crucial data, or theories. In fact, change in how one decides in favor of one conception over another is a complex part of conceptual development (Kuhn & Phelps, 1982; Schauble, 1990), and confrontation is an implausible mechanism for changing principles that decide, for example, the relevance of data to theory.

Confrontation is also problematic as an instructional model. In contrast to more evenhanded approaches to classroom discussions in which students are encouraged to evaluate their conceptions relative to the complexity of the phenomena or problem, confrontation essentially denies the validity of students' ideas. It communicates to students that their specific conceptions and their general efforts to understand are fundamentally flawed. The metaphor of confrontation is also inconsistent with the pedagogical sensitivity and care required to negotiate new understandings in the classroom (Yaekel, Cobb, & Wood, 1991). Finally, some misconceptions are powerful enough to influence what students actually perceive, thereby decreasing the chances that planned confrontation and competition will be successful (Resnick, 1983).

We have reached a transitional point in our analysis. With the key assertions of misconceptions research and some of its problems before us, we can begin to build toward a more productive theoretical perspective. First, we present three case analyses from our work in learning mathematics and physics that emphasize continuity between naive and advanced states of understanding. They provide important clues to how students' prior knowl-

edge can support the gradual acquisition of expertise—that is, how construc-
tivist learning is possible.

WHO THINKS CONCRETELY?

Misconceptions and alternative conceptions approaches to science learn-
ing have taken students' conceptions to be similar in form to expert
knowledge but different in content. On the other hand, researchers work-
ing within the expert–novice paradigm have focused on apparent differ-
ences in form between students' and experts' knowledge. They have
characterized novices' reasoning as concrete and that of physics experts
as abstract (Chi, Feltovich, & Glaser, 1981; Larkin, 1983). Learning
Newtonian physics means leaving the everyday world of concrete objects
and entering the abstract world of physics through direct instruction in
experts' concepts. Abstraction constitutes the principal barrier that di-
vides experts from novices—a barrier surmounted only when novices
gain new and fundamentally different knowledge of the physical world.
This point of view therefore denies the relevance of novice conceptions
to the heart of experts' competence.[7]

Being concrete means being focused on the everyday and obvious surface
structure of physical problem situations. So, novices classify problems in
Newtonian mechanics that contain pulleys as pulley problems whether they
are solved by $F = ma$ or, alternatively, by conservation of energy (Chi et al.,
1981). Being solvable by one or another physical principle defines catego-
ries of problems only for physicists, and these categories relate to the
abstract, deep structures that physics instruction provides.

Our analysis suggests that the intuitive notions of physics novices contain
both a sense of surface structure and a sense of deep structure. The deep
structures of intuitive physics are in no obvious way less abstract than those
of schooled physics. They may be more familiar, but they do not relate to
classifying problems by the familiar objects (e.g., pulleys) they contain. The
reasons novices appear less abstract are largely methodological. The deeper
structure that novices perceive is not normally tapped in the assessments of

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[7]There are at least two reasons for introducing a distinction more directly associated with
expert–novice studies than misconceptions research per se. First, continuity from naive to
expert ideas is the basis both for our critique of misconceptions and for the constructivist
theoretical proposal later in the article. To ignore a commonly cited dimension of difference
between naive and expert knowledge would leave our argument open to the criticism that we had
successfully questioned some distinctions but left the strongest arguments for discontinuity
intact. More deeply, there is great commonality between the misconceptions perspective and
expert–novice studies—specifically that naive ideas are unlikely resources for acquiring exper-
tise, and hence, instruction of new concepts is a necessity—that adds breadth to our argument.
It is worth pointing to the main source of those commonalities: inattention to the positive
characteristics of naive knowledge.

expert–novice studies. Novices are asked to perform under conditions in which they are incompetent, and the questions they answer are inappropriate measures of their real competence. They are characterized as bound to the concrete, conceiving only the directly perceivable aspects of situations. Our alternative position is that there are many abstract elements in intuitive physics knowledge. People see processes like bouncing and falling that are independent of many surface features, and therefore, must count as abstract. (A bouncing pulley is a lot more like a bouncing inclined plane, in naive and obvious ways, than a bouncing pulley is like a floating one.) Even when situations are difficult, but still close to their intuitive competence, novices can reason abstractly using knowledge that is similar in form to that of experts. To see this, one must get a sense for the kind of abstract knowledge novices possess, the kind of problems that engage it, and the sorts of inferences it supports.

We show that novices can exhibit expert-like behavior in explaining how a complex but familiar physical system works. Specifically, novices are willing to search for appropriate underlying mechanisms that are independent of salient surface representations. The heart of this analysis is that explanation is an everyday activity. People know that the world operates according to general principles and that those principles apply sometimes in highly nonobvious ways. We constantly explain things to ourselves and to each other. The terms we use in these explanations are often not evident in the situation and are in no clear way less abstract than physicists' terms. Put somewhat differently, if people had only concrete knowledge, everyday explanations would make no sense as an activity.

Our method for highlighting the abstractness of intuitive physics is to put novices in problematic but familiar situations. There we will see hallmarks of abstract reasoning, including hypothetical reasoning, collecting evidence for the presence or absence of some abstract entity, and drawing conclusions from general principles. We claim that thinking about support, clamping, tension, pushes, and pulls are not surface descriptions of the physical world, no matter how quickly we recognize them in physical situations.

Reasoning About Bicycles

As a classroom exercise with physics-naive education graduate students, one of us (diSessa) evolved an extended discussion and analysis of how bicycles work. If novices are concrete, surely these students should show their concreteness in such a familiar situation. The opening question was to ask simply how the bicycle's frame is supported—why does it not fall to the ground? After some encouragement that the interrogator was serious, or after someone proposed a controversial explanation, the question was accepted as not only meaningful but also interesting. Students do not just accept the obvious fact that bicycle frames simply do not fall to the

ground. Like experts, they realize the need for deeper explanatory principles than superficial descriptions of the structure of the bicycle.

The intuitive concept of support.   Students frequently begin by articulating the need to find the path of support from the ground to the frame of the bicycle. The *propagation of support* is, of course, a familiar idea that explains why the top book of a pile sits high in the air, yet its burden is felt by the books below, and why a scale under a pile of books feels the weight of a newly added book. That the concept of support is familiar hardly qualifies it as concrete. The very fact that the answer to this question of support for the bicycle is problematic implies that students do not always expect support to be visually evident. In the same way, physicists do not expect to literally see momentum propagation, even if they are quick to know momentum is propagating in some situations.

There are many kinds of support. Determining which operates in a given situation may require as much extensive reasoning for novices as for experts puzzling over whether a problem will yield to one principle or another. Support is also intuitively a topological and causal issue in roughly the same way that Newtonian statics views it. Nearly the same set of superficial properties is stripped away to find the relevant deep structure. Color, texture, history, ownership, temperature, and so on, are irrelevant; flexibility, rigidity, stress, and contact are the deep structure of support.

Typically, students' first conjecture is that support is provided by the spokes directly under the hub, similar to the pile of books under the top one. How concrete is this attribution? A good indication is that the claim is often immediately withdrawn if it is pointed out that the spokes are under tension. The bottom spokes are pulling down on the hub, and the existence of tension, which is equally hidden from direct perception, contradicts presumptions of support. Some students know for a fact that the spokes are under tension; others need to be convinced. That argument can involve other considerations. For example, if you pluck a bicycle spoke, it vibrates like a rubber band under tension. This observation may initially need to be made by the interrogator, but once introduced, it is generally taken as compelling evidence of the existence of another imperceptible structure—tension.

Hypothetical reasoning.   Sometimes students propose (or respond positively to) a hypothetical argument against the support proposal: If substantial pressure were put on the spokes, they would crumple. Therefore, they must be under tension, or at least are very unlikely to bear much burden. How concrete is the spindliness that students see in spokes? If it is a surface feature, why do they not immediately see it and disregard the possibility that the spokes may support the hub and frame?

Reasoning by exclusion.   A pause in the flow of ideas usually occurs at this point. By ruling out the previous possibilities, students are eventually left with the hypothesis that the hub of the bicycle is hanging by the top spokes. Initially, this is a highly counterintuitive attribution. Still, given time to consider the kinds of arguments described previously, most come to feel it is a forced conclusion. Intuitive reasoning, therefore, can draw initially implausible conclusions because intuitive knowledge is not so shallow or weak that it cannot flexibly support new ways of conceptualizing everyday situations. This is hardly the characteristic of a concrete, surface feature-bound knowledge system.

Other intuitive abstractions.   Other intuitive descriptions that often apply hypothetically in the physical world are invoked by students in the bicycle situation. The first is an alternative to the hanging theory of support: The hub is being locked into place by the spokes, which are pulling in all directions at once. This analysis, again, involves a highly nonevident causal attribution, though a familiar one. Pairs of opposing forces (i.e., two hands pushing or pulling equally on a block) seem to clamp objects rigidly in place. A vise is a device that depends dramatically on this mechanism. Opposing pulls on manacles restrain a prisoner. Yet, familiarity cannot be used to argue that such mechanisms are concrete, or any principle is considered concrete if it has a familiar instantiation. Does the image of a man jumping forward in a boat, which physicists instantly recognize as a case of action and reaction, mean that Newton's third law is a concrete principle?

In what ways are the spokes of a wheel like hands clamping an object? They are the same in that opposing, equal tendencies impinge on a common object. Notice how much this characterization strips from the surface presentation. It removes not only all the attributes that we intuitively understand do not contribute to motion or rest, but also even the animacy of the cause. It is irrelevant that people restrain a prisoner and that spokes may restrain a hub. People can see the hub to be like clamping hands only by characterizing each as geometric configurations of invisible causal tendencies such as pulls and pushes. Even to see a pull (tension) in the spoke is quite an abstract accomplishment because the hub never moves in response to that pull.

We take one final example from the bicycle. Leaving aside the spokes, how does the bicycle tire support the rim? The intuitive model most frequently proposed by students is that the compression of the tire causes an increase in the internal pressure, which pushes up on the rim. The tire is then a kind of spring. Again, how concrete is this? And how tied to surface features of the situation? How can students possibly suggest, let alone judge, the metaphor *like a spring* unless they have an abstract model of how a spring works?

Unfortunately, the spring model fails for the tire. The pressure in the tire increases only minimally in compression, not nearly enough to account for support. Even worse, the increase in pressure cannot be responsible for

support through a net increased upward force on the wheel because air pressure propagates around the inside of the tire and presses down on the top of the rim as much is it presses up on the bottom. Yet, people have abstracted from experience the principle that compression leads to increased pressure, and increased pressure can provide more support. It is exactly this abstract model that explains how a bicycle tire may be thought of as a spring.

Abstract and concrete are slippery terms. We believe they are far from sufficiently precise to adequately classify knowledge, let alone to classify the people who use that knowledge. Experts appear highly concrete when they compulsively turn even the most abstruse ideas into a most concrete set of visible and manipulable entities—algebraic symbols. Concretely mediated thinking of this sort is characteristic of the highest levels of human activity. Similarly, novices are abstract—in addition to the ways suggested previously—when they apply newly learned physical concepts to situations without examining all the details of those situations that tell a physicist those concepts may not be relevant. So, it is especially important to be clear about what is asserted in the name of abstract and concrete.

We claim that novices in reasoning about the physical world:

1. Seek deeper explanations of the causality involved in situations than are immediately and superficially apparent.
2. Attend extremely selectively to features of situations, ignoring (abstracting from) many surface features to focus on what they consider causally relevant.
3. Apply principles that (a) apply hypothetically to a given situation, (b) are intended to identify underlying causal mechanisms (deep structure), and (c) may be withdrawn under consideration of other arguments.

In short, when a complex fabric of physical relationships, potential observations, and interventions (like plucking a spoke to find out if it is under tension) mediate novices' determinations of how situations are causally configured, we believe it is appropriate to say such knowledge is abstract in the same sense that expert knowledge is abstract. Novices appear to think concretely when they have been asked to classify problems that they are unable to solve and have nothing but generic, noncausal descriptions to rely on. Experts are classified as abstract because they have the particular abstractions selected as relevant— those that happen to solve the problems posed in research studies.

SHARED CHARACTERISTICS OF KNOWLEDGE SYSTEMS: NOVICES AND MASTERS OF COMMON FRACTIONS

As the earlier references to mathematical misconceptions have shown, much recent research in mathematics learning has focused on how students' partial

and evolving understandings differ from those of experts. This difference has been emphasized in analyses of many domains, including order relations among decimal fractions (Hiebert & Wearne, 1986, Nesher & Peled, 1986; Resnick et al., 1989), decimal fractions operations (Fischbein et al., 1985; Hiebert & Wearne, 1985, 1986), and basic notions of sampling, chance, and probability (Shaughnessy, 1977, 1985, 1992; Tversky & Kahneman, 1982). These analyses have generally asserted that the flaws in students' understandings result from overgeneralized applications of prior mathematical knowledge—for example, using only knowledge of whole number order and place value to order decimals. Although many students eventually work through and beyond their flawed conceptualizations, mastery of these elementary mathematical domains is neither easy, rapid, nor uniformly achieved (Post, Harel, Behr, & Lesh, 1988; Tversky & Kahneman, 1982).

This case summarizes a recent study of student learning in one such domain: order and equivalence relations among fractions (Smith, 1990). Whereas the previous analysis questioned a distinction commonly used to distinguish experts from novices, this case points directly to properties that are descriptive both of the knowledge of masters of the domain and of novices. It focuses on the shared general characteristics of masters' and novices' knowledge systems, on masters' reuse of novice knowledge, and on shifts in the application context of key pieces of knowledge as an important category of learning. It suggests that these continuities may be important general dimensions of conceptual growth in a constructivist theory of mathematics learning.

The study compared the knowledge and reasoning of upper elementary, middle school, and senior high students on various tasks involving the order (>) and equivalence (=) of fractions—issues central to a broader understanding of rational numbers (Behr, Wachsmuth, Post, & Lesh, 1984; Hiebert, 1992). Students were asked to compare pairs of fractions, select two addends for a given sum from a fixed set of possibilities, generate fractions between two given fractions, and evaluate correct and incorrect examples of comparison reasoning. Their knowledge was assessed in terms of the strategies they used to solve the tasks. For example, a frequent strategy for comparing 8/11 and 7/15 was to assert that the common reference number, ½, was between 8/11 and 7/15, and to use that order relation to infer that 8/11 is greater than 7/15. Ten elementary students who had just been introduced to fractions in the classroom were taken as novices. Eight senior high students and three middle-school students whose reasoning was accurate, direct, flexible, and confident were judged representative masters of the domain.

Fraction Knowledge in Pieces

Two strategies for reasoning about fractions are sufficiently general to solve most (if not all) problems in the domain—conversion to common denominator and conversion to decimal. When the fractions given in a

problem are converted in either way, it can be solved by applying appropriate whole-number or decimal knowledge. It is, therefore, not surprising that most textbook curricula, particularly in the middle-school years, emphasize these strategies almost exclusively as the appropriate methods for reasoning about fraction order and equivalence.

Because one trademark of mathematical knowledge is its generality and because general strategies are heavily emphasized in instruction, masters might have been expected to rely solely on these strategies in their solutions. In the same way that $F = ma$ and conservation of energy dominate experts' solutions of typical mechanics problems, common denominator and decimal conversion may dominate masters' fraction reasoning. In fact, those general strategies were not so widely applied as expected. Mastery depended instead on a wider variety of strategies, many of which are applicable to only a restricted class of problems. Many of these specific strategies were not supported in any direct way in the textbook curriculum.

Within the restricted class of problems they could solve, these strategies supported very efficient and reliable solutions. Although one of three strategies was sufficient alone to solve all eight comparison problems,[8] masters applied an average of 7.6 different strategies.[9] This varied strategy use was also relatively uniform across the 11 masters. Only 1 student used the same strategy on as many as four different comparisons, and only 2 others did so on three different comparisons. The diversity of strategy application was greatest on the comparison task (which also contained the largest number of items among the tasks) but was evident on the other tasks as well.[10]

Let us look at some specific strategies. One general resource used by masters was well-known numerical *reference points* (Behr et al., 1984). Because the four tasks involved only proper fractions, the relevant reference points were 0, ½, and 1. When they compared 8/11 and 7/15, about half of the masters applied the strategy described previously (8/11 > 7/15 because 8/11 > ½ and 7/15 < ½). Likewise, when they searched for addends for 5/6 (the addend choices were ½, 1/3, ¼, and 1/6), they used the fact that 5/6 was close to 1 to motivate their selection of the biggest addends—½ and 1/3. Both strategies produced rapid and reliable solutions without resort to time-consuming numerical computations.

Similarly, when they compared 12/24 and 8/16, masters exploited the easy *half of* relationship within the components of fractions to assert their equivalence. They again avoided the more time-consuming computations required

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[8]In addition to conversion to common denominator and conversion to decimal, cross multiplication also solves all fraction comparisons.

[9]Frequently, subjects applied more than one strategy to solve a single item.

[10]The clear exception was reasoning about the betweenness. The fractions given on that task, 3/5 and 5/7, were sufficiently close to each other so it was difficult to solve the task without applying a general strategy. Students' solutions reflected this situation; generally speaking, they either converted to common denominator or decimal or failed to solve the task.

by either decimal or common-denominator conversion. Like the two previous strategies, the efficiency of this strategy depended on its limited applicability to specific numerical situations. Although it was deployed somewhat more generally by some masters (e.g., to compare ¼ with 6/24 and 4/6 with 6/9), the payoff in efficiency and reliability is purchased at the cost of generality. Instead of exploiting the power of general strategies to reason about all fractions they confronted, masters more often used specific tools that were well suited to particular numerical situations, leaving the general strategies for situations in which there were no easy relationships to exploit.

Despite their utility and efficiency, most specific strategies are not explicitly taught. A review of representative textbook series conducted as part of this study revealed that very few of the specific strategies were supported in any direct way in the Grade 3 through eight texts. Thus, the origin of these specific strategies appears to lie only indirectly with instruction and must substantially involve the constructive activity of students.

Given this view of mastery, how do novices in the domain compare? Despite limited exposure to instruction, the elementary students also applied a variety of strategies to solve the tasks. In terms of numbers of strategies applied, they rivaled the masters, averaging of seven strategies to solve eight comparison items. Only one student used the same strategy on all eight items. Their reasoning was generally organized around the mental or graphic manipulation of drawings of divided quantities (e.g., pie charts). The fraction 2/3 was represented as the fractional quantity formed by selecting two of three equal-sized parts of a rectangular or circular whole quantity. This general model supported the development of a variety of specific strategies. Moreover, the same general consideration—the specific numerical features of the given fractions—was a crucial feature of novice strategies. Novices, like masters, used the half of relationship to establish the equivalence of 12/24 and 8/16 and generalized that strategy somewhat to equate ¼ and 6/24 (fourth of) and 4/6 and 6/9 (two thirds of). Masters and novices both possessed varied resources and the disposition to select strategies that were maximally effective in restricted numerical contexts.

These results indicated a fundamental similarity in the systematic characteristics of masters' and novices' knowledge. It consisted of numerous strategies, groups of which were related by a common terminology (e.g., standard reference numbers and parts of divided quantities). Neither group's reasoning was captured very well by the general strategies typically emphasized in textbooks. There was a shared tendency to construct strategies that were tailored to solving specific classes of problems. Whenever possible, most novices and masters preferred to apply specific strategies, leaving the more cumbersome general strategies for those situations in which no specific strategy was effective. It is important to note that similar characterizations have been given for children's knowledge of natural number addition and subtraction (Carpenter & Moser, 1984; Murray & Olivier, 1989; Siegler, 1987; Siegler & Jenkins, 1989).

It would be a mistake, however, to discount the important differences between two groups. Mastering this domain is a complex achievement, requiring substantive conceptual changes over a period of years. Novice reasoning was often narrowly restricted to the manipulation of a mental model of divided quantity. In contrast, masters' reasoning was carried more efficiently on three different types of numerical relationships: (a) relations within and between the whole-number components, (b) numerical reference points, or (c) numerical conversions such as common denominator and decimal. From this description, masters' knowledge may appear to be quite different in content from novices' mental model of divided quantity, but knowledge of divided quantity played an important conceptual role for masters just below the numerical surface of their reasoning.

Continuity in Knowledge Content: The Root of Divided Quantity

On the surface, masters and novices appeared to be doing quite different things. Masters solved the tasks with a variety of different numerical strategies, whereas novices depended directly on their models of quantity, but a careful examination of masters' reasoning indicated that their numerical strategies were built on a foundation of divided-quantity knowledge. This foundation became most visible when masters were asked to justify their numerical reasoning or when they encountered some difficulty in solving a problem. When they were asked why their numerical strategies worked, particularly those that were stated in terms of relations between fraction components, masters generally justified their reasoning in terms of divided quantity. For example, masters frequently solved the comparison of 7/11 and 7/13 by asserting that "a smaller denominator makes the fraction greater," provided that the numerators are equal. When asked to explain that assertion, they appealed to the size of the parts in the respective models of quantity—that "11ths are larger size pieces than 13ths." The latter part of the argument is typical of novice reasoning. Likewise, when masters were asked to justify why conversion to common denominator worked, they most frequently appealed to the fact that the quantities expressed by the fractions were invariant under the subdivisions of their parts. Their justifications involved coordinating the numerical steps in the conversion (e.g., raising terms by multiplication) with corresponding actions on divided quantity (subdividing the parts in the quantity equally).

This changing role of quantity-based fraction knowledge represents a different form of continuity between novice and expert. Prior novice knowledge remained productive for masters by supporting new knowledge that was more efficient and reliable. This shift involved a change in role of their knowledge of divided quantity. When divided quantities carried the reasoning of many novices directly, masters' reasoning was carried by numerical relationships and

operations. Their knowledge of divided quantity, though often not directly apparent, remained at the basis of their knowledge system, underlying and justifying their more powerful numerical strategies.

This analysis of learning is quite different from the misconceptions view. Prior knowledge of divided quantity is certainly not replaced by the general strategies emphasized in textbook curriculum. General strategies are definitely learned by masters and play a major role in their reasoning, but they do not displace the strategies that precede them. In fact, evidence suggests the opposite possibility: Learning the general strategies themselves may depend on prior knowledge of divided quantity. Instead of being displaced, prior knowledge is retained and serves a new foundational role in developing mastery.

Consistent with the basic premise of constructivism, this case illustrates specific ways that students utilize their prior conceptions to learn more advanced knowledge. It suggests that researchers look for ways in which knowledge is reused and serves new functions in developing competence. It also serves as a reminder that surface differences between novices and masters can hide important similarities in the content of their domain knowledge. If no attempt is made to probe below the surface of the experts' knowledge, important genetic connections to prior states can be overlooked. The result can be seriously flawed characterizations of expertise and processes involved in learning complex mathematical ideas.

Continuity of Form: Adjusting the Application Context

The previous examples highlight two important dimensions of continuity between masters' and novices' fraction knowledge: (a) Neither group's knowledge was simply composed but was instead structured as complex systems of related elements, and (b) a major component of novices' understanding was retained to play a different role for masters' understanding. A third dimension of continuity focuses on changes in the applicability of knowledge—learning to use what you already know in either wider or more restricted contexts.

Some novices knew that the strategy of common-denominator conversion worked in the addition context. Five of the 10 students were able to correctly add two fractions with unlike denominators (i.e., 3/5 + 7/10), even though they could not coherently explain the necessity of converting 3/5 to 6/10. Of those 5, only 1 applied that strategy in the comparison and betweenness contexts. In contrast, conversions to common denominator were central to masters' solutions to the task of finding fractions between 3/5 and 5/7 and to difficult comparisons. Six of the 11 used it to solve the betweenness task, and 7 applied it on at least one comparison item.

General strategies like conversion to common denominator are powerful tools, but that power is generally harnessed incrementally, rather than all at once. Although extensive data were not available on this issue, one striking example illustrates this point. A middle-school subject, KS, used common-denominator conversion to find addends but failed to do so on either compar-

ison or betweenness problems. He struggled to compare 3/5 and 5/7 and could not solve the betweenness task involving those same fractions. At the end of the interview, he was shown how common-denominator conversion solved the comparison. When he indicated that he understood and appreciated that solution, he was again asked if there were numbers between 3/5 and 5/7. He replied that he did not think so and gave no indication that he saw any relation between the comparison he had just solved and the issue of betweenness. Consistent with this telling example, nonmasters generally did not immediately follow up one extension of the applicability of common-denominator conversion with another.

A related example involves the strategy of reasoning about the size of the denominator in the comparison context. The Denominator Principle strategy asserts that the fraction with the smaller denominator is greatest, provided that the numerators are equal. Several novice students drew incorrect conclusions using the divided-quantity version of this strategy because they considered only the size of parts indicated by the denominator in comparing fractions. For example, they ignored the effect of the numerators and concluded that 4/6 is greater than 6/9 because "6ths are larger than 9ths." Their strategy was conceptually correct if incomplete (smaller denominators do indicate larger parts and therefore tend to increase fraction size), but they had not restricted its application to equal-numerator situations. Masters used the same strategy but only when they were sure that they could ignore the effects of the numerators, a restriction of context that was, in some cases, more sophisticated than it appeared. All masters learned to apply the strategy in the equal-numerator case, but some went on to extract more power from it by relaxing the standard constraints a bit to include cases in which the numerators were approximately equal. This generalization allowed some to conclude, for example, that 8/11 is greater than 7/15 because 8 is approximately equal to 7 and 11 is greater than 15.

We claim that learning in both of these cases involves shifts in the applicability of strategies more than changes in the content of the strategies themselves. The examples suggest that mastery is achieved, in part, by using what you already know in more general and powerful ways and also by learning where and why pieces of knowledge that are conceptually correct may work only in more restricted contexts.

ROLES FOR PRIOR KNOWLEDGE IN
SCIENTIFIC REASONING

In this final case we return to the domain of Newtonian physics and argue that everyday physical conceptions are centrally involved in expert scientific reasoning. The main goal of the case is to illustrate three specific roles for prior knowledge in scientific expertise: providing raw material for formulating scientific theory, supporting qualitative reasoning, and mapping every-

day situations to theoretical representations. Our argument for the first role is historical.

The Development of Scientific Theory

Einstein (1950; Miller, 1986) proposed a view of scientific knowledge that emphasized the connection between experience and primitive theoretical concepts. Everyday reasoning supplies a huge store of abstractions of physical experience that are useful in everyday life. From these everyday ideas, the scientist builds a set of axioms. For Einstein, the process of axiomatization, not induction, was the core of scientific practice. Through this process everyday intuitions are altered in character and structure, becoming the rigorous foundations of a deductive knowledge system. They must then conform to scientific criteria of consistency, coherence, and completeness—criteria not characteristic of everyday knowledge.

There are numerous examples in the history of the physical sciences that illustrate this process. Einstein's special theory of relativity evolved from the consideration of a practically rigid rod and the use of this rod to measure objects. Einstein emphasized the nature of the rigid body concept as a prescientific notion closely related to everyday intuitions about space and time. (Indeed, he discussed with Piaget the possible connections between his own thinking and children's spatial reasoning.) His theory was generated by refining those intuitions into a rigorous system of axioms. Likewise, the *curls* and *divergences* in Maxwell's equations for electromagnetism originated as a grid of whirlpools in a fluid separated by ball bearing idle wheels. Newton related his particle theory of light to the motion of tennis balls. In emphasizing that new theories emerge from clear conceptions rooted in other, older conceptual systems, Maxwell suggested (Lightman, 1989),

> We must, therefore, discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical [empirical] science from which that conception is borrowed. (p. 100)

In each of these examples, theory development required selecting the right piece of everyday physical knowledge to reformulate as a core theoretical concept. Reformulation changed the systematic features of the concept by embedding it in a formal theory, but that did not change the fact that the concept began as an everyday idea.

Supporting Qualitative Reasoning

Research conducted within the expert–novice paradigm has emphasized how experts reason with concepts different from those of novices (Chi et

al., 1981; Larkin, 1983; Reif, 1985). But if the view that everyday experience is refined and reused in scientific thinking is correct, we should find some use of everyday ideas in the reasoning of experts. We attempt to make this connection by reanalyzing an example of expert reasoning taken from Larkin's (1983) analysis of expert and novice physics problem solving.

Larkin's analysis emphasized fundamental differences between novice and expert reasoning. She claimed that her expert and novice subjects used different representations and different concepts. Without disputing that experts' reasoning is different in important ways from that of novices, we emphasize the substantial continuities between them. We also highlight areas in which Larkin's theoretical analysis is murky, in which experts' and novices' representations and concepts—at least as seen in this example—do not seem very different at all. In this case as before, we find it difficult to characterize the important differences between experts and novices in terms of the simple and traditional dimensions of discontinuity: concrete versus abstract, familiar versus theoretical, or surface versus deep.

The problem situation is depicted in Figure 1. An expert and a novice were each asked, "What constant horizontal force F must be applied to the large cart (of mass M) so that the smaller carts (masses m_1 and m_2) do not move relative to the large cart? Neglect friction." The excerpts from their solutions that were quoted by Larkin are reproduced.

Novice: Well, I'm right now trying to reason why it isn't going to move. I mean I can see, if you accelerated it at a certain speed, the wind would push on m_1 so m_2 wouldn't fall.
(later)
Once I visualize it, I can probably get started. But I don't see how this is going to work.
Expert: Well, with a uniformly accelerating reference frame, all right? So that there is a pseudoforce on m_1 to the left. That is just equivalent—just necessary to balance out the weight of m_2. (p. 81)
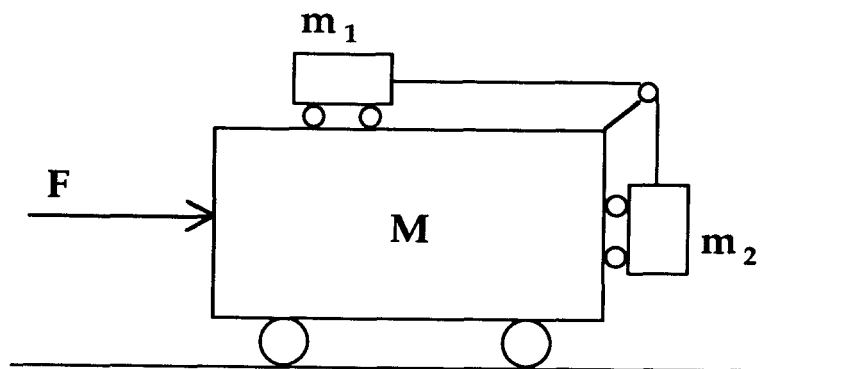

FIGURE 1   Sketch shown to novice and expert.

Representation.   Larkin analyzed the expert and the novice solutions in terms of the differences between their representations of the problem. "[T]he *physical* representation used by the skillful experts seems profoundly different in content from *naive* representation used by the novice subjects... " (p. 81). On this account, the expert solved the problem by constructing a problem representation that contained only abstract physical entities and avoided commonsense reasoning. Physical entities are the constructs of Newtonian theory, such as forces, accelerations, and momenta. The novice, on the other hand, formed a representation containing concrete *familiar* entities, which led him or her to see "a confusing collection of carts and ropes and pulleys" (p. 81) and impeded progress toward a solution. Larkin's account is consistent with the misconceptions view of learning; if students are to become scientists, they must replace their prior common-sense physical conceptions with the concepts of abstract physics.

It is possible, however, to give a different reading of these two examples that includes some important parallels between the novice and the expert. Like Larkin, we focus on each subject's problem representation. The novice apparently felt the contradiction between his or her intuition that the little blocks should move down and to the right, and the stated information that the blocks should not move relative to the cart. To resolve it, he or she introduced a hypothetical influence—the wind—that would push toward the left on the top block. This move is a plausible application of everyday physical knowledge. The wind is a force that can occur naturally in this situation, and it can balance the force of gravity and prevent the block m_2 from falling. We have depicted the inferred novice representation in Figure 2 by drawing in the force of gravity and the force of the wind. If these forces balance, the small carts will not move.

The expert likely recognized the same problem, but instead of appealing
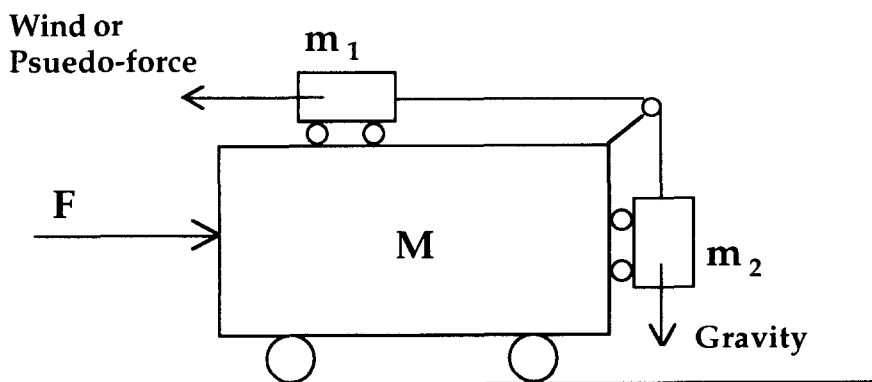

FIGURE 2   Novices' (and experts') problem representation.

to wind resistance, he or she invoked the concept of an accelerating frame of reference. According to Larkin, this action is equivalent to constructing a scientific representation. However, a plausible alternative interpretation is that the accelerating frame of reference merely justified the transformation of one common problem situation into another. In the given situation, the big block is moving, and there is only the tension force on m_1. In the transformed situation, the big block is stationary, and there is an additional leftward pseudoforce on m_1. The important point is that this second situation is not more scientific, theoretical, or abstract; it is simply another common situation. For example, the transformed situation is isomorphic to the problem of holding a bucket at the top of a well by pulling horizontally on its rope. The recognizable physics concept—accelerating frame of reference—serves only to justify the transformation.[11]

An important part of expertise is the ability to transform a wide range of problematic situations into a smaller number of more familiar and unproblematic ones. In this case the transformation involved thinking of the given problem as a situation in which the cart is at rest, but with the additional force to the left on m_1 (i.e., the pseudoforce). This representation can be depicted exactly as the novice's representation (Figure 2). Thus, the novice and the expert may be constructing essentially the same representation—though with different conceptual components—in which a force to the left on m_1 balances the downward gravitational force on m_2.

Concepts.   The same conceptual schema, *balancing,* appears to play a crucial role in both the expert's and the novice's reasoning. Balancing is an abstract piece of intuitive physical knowledge that requires the correspondence of two or more elements in a system to establish an equilibrium (diSessa, 1983, in press; Johnson, 1987). It leads both novice and expert to presuppose the existence of a leftward force on m_1. The expert explicitly mentions balancing, "the pseudoforce on m_1 is ... just necessary to balance out the weight of m_2," and uses it in his or her explanation, "the existence of the tension force and fact that the acceleration of the cart is zero (relative to the large cart) implies that there is some force directed to the left" (Larkin, 1983, p. 82). Although balancing is not a core concept in Newtonian physics, here it stands in for that core and does essential work in solving the problem. The novice does not explicitly mention balancing, but his or her line of reasoning follows the same basic scheme: Despite the presence of gravity on m_2, m_2 does not fall. Therefore, there must be another force, such as the wind on m_1, to maintain the stationary state.

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[11] diSessa (1993) focused explicitly on revised justification structure as a fundamental system parameter differentiating novices and experts.

Traditional dimensions of difference?   Because we argue that the expert and the novice apply the same concept—balancing—to essentially the same problem representation, it might appear that we wish to equate their reasoning. This is not the case. The important differences between their solutions lie in details of how they connected balancing to the information given in the problem and in how accountable their solutions were to Newtonian theory. But before examining these differences, we consider the possibility that the differences are matters of abstractness, depth, or rigor.

Both novice and expert link their analyses to familiar experiences. The novice explicitly generates a hypothetical force—the wind—that bridges between his or her experience and theory. Although the expert did not explicitly link the pseudoforce to sensory experience, Larkin herself provides this connection: "the so-called pseudoforce is the 'force' that you feel snapping your head back in ... a car starting quickly from a stoplight" (1983, p. 81). It is hard to see how the expert's pseudoforce, as characterized by Larkin, is any more abstract than the novice's wind. Both the wind and the whiplash force are constructions inferred from their effects.

Both expert and novice also rapidly simplify and reformulate the problem, producing a deep analysis of the situation. Neither exhaustively lists all the forces in the situation—such as the normal forces exerted by surfaces or the tension forces exerted by the strings, nor do they mention the redirection of the tension forces by the pulley. Considering every force acting in the cart system and properly solving the resulting equations would have led to a much more cumbersome and time-consuming solution. Instead, both focus on exactly two forces, the weight of m_1 and the retaining force exerted on m_2. This simplification of the problem is warranted both by our experience with pulleys as things that redirect our pulls and by the typical textbook treatment of pulleys as mechanisms that redirect forces. But this simplification is neither capricious nor transparent; it indicates a high-level parsing of the confusing collection of ropes and pulleys into an functional pulley system. The pulley system indeed has an internal structure of many components, but its behavior is entirely predictable from the two mentioned external forces. Thus, both novice and expert seem to have used concepts available from everyday knowledge (pulley systems) in place of a thorough and exhaustive analysis of the forces involved. Both their solutions represent a selection of deep features that cut to the physical heart of the problem.

Another apparent difference between novice and expert is their appeal to rigorous theory, but here again, the difference may be smaller than it seems. In our view, the expert's use of technical terms (e.g., *accelerating reference frame*) is far from an example of rigorous scientific reasoning. In Larkin's quoted excerpts, the novice and the expert are on equally shaky grounds in terms of their degrees of scientific care, completeness, and justification. The novice seemed aware that his or her introduction of the wind was a questionable move. The expert's move, however, was only barely more legitimate. Specifically, the expert used an inertial concept (balancing forces) in a

noninertial (accelerating) reference frame. This is precisely the same as using centrifugal forces to analyze orbits, a mistake that would certainly be marked wrong on a college physics exam. Physics teachers who are careful to distinguish textbook physics from pop physics consistently deny the legitimate use of centrifugal and other pseudoforces.[12] But experts regularly engage in such sloppy reasoning when the intuitive ideas are useful and when their use does not undermine the possibility of developing more careful analyses, if necessary.

Where are the differences?   The wind force proposed by the novice is different from the pseudoforce proposed by the expert. The former is problematic in light of the problem statement (neglect friction), the sizes of carts and forces likely in such a situation, and the lack of information necessary to compute air resistance. Moreover, although both novice and expert reason from a representation in which the big block is stationary, the expert knows that the transformation from an accelerating to a stationary frame of reference has consequences for his or her analysis. We expect that the novice was unaware of these consequences. Thus, although neither the representations nor concepts were particularly different, the relationships between specific elements and concepts were markedly different. We expect these differences would allow an expert to smoothly elaborate the quoted "quick and dirty" analysis into a more complete analysis framed more carefully in Newtonian terms. In general, we expect an expert to have a large collection of techniques available to reduce complex situations to simpler ones without introducing errors. In this situation at least, the novice lacks a simplification technique that permits a rapid, justifiable qualitative solution.

Mapping Physical Situations to Core Theory

Although physics problems may appear well defined, the process of applying Newtonian physics to familiar situations is a complex process (Chi et al., 1981). The skills for coping with this complexity are buried in experts' tacit knowledge and are not ordinarily discussed in physics classes. Chief among them is the process of problem formulation: mapping physical situations to appropriate theoretical models. The complexity of mapping situations to theories ultimately arises from the gap between the complexity of the real world and the sparseness of Newtonian theory. Consider, for example, a

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[12]It is possible to define pseudoforces formally, but they are not the same class of forces that define Newtonian interactions. No formal treatment would bother to introduce pseudoforces; they are unnecessary and a distraction, except that they allow transforming some situations into more familiar ones.

rubber band. In order to reason about physical situations involving rubber bands, one must map the rubber band to some entity in a theoretical model. This mapping, however, is extremely context dependent. With a rubber band, one can: (a) unite the separate sections of a newspaper into a rigid body that can be thrown, (b) make a solid object attached to one end bob up and down, (c) create a musical stringed instrument, (d) open a tight jar lid, (e) store energy for a toy plane's propeller.

For each of these functions, the rubber band corresponds to a different scientific entity. In the first case, one would likely not model the rubber band at all, but simply consider the rubber band/newspaper combination as a single point mass. In the second, one would model the rubber band as a interaction force governed by Hooke's spring law. In the third, one might again think of Hooke's spring law, but this time in terms of the side-to-side vibrations of the rubber band rather than the vibrations along its length. In the fourth case, what is important about the rubber band is its coefficient of friction, and in the fifth, its energy-storing torsion. The ability to form the appropriate scientific representation of any problem situation involving a rubber band, therefore, depends on an analysis of its function in each situation.

In analyses of physics expertise, the knowledge involved in coordinating problem situations with scientific models has often been described in terms of applicability conditions (e.g., Chi et al., 1981; Reif, 1985). One might ask how such conditions can be stated. One initial possibility is that applicability is simply a one-to-one mapping of physical objects and properties to theoretical entities. In celestial mechanics, a moon is mapped to a point mass. However, this proposal falls apart for the rubber band, which maps one-to-many to scientific terms, depending on the problem situation. Another possible approach is to make topological distinctions, but in the newspaper and jar examples, the rubber band is wrapped around an object, whereas in other cases, it is attached only at two points. The single topological description, "attached at two points," would then still map to three different applications of scientific laws. So the mapping between problem situations and the various Newtonian models is difficult to state in either context-free or topological terms.

In contrast, a set of everyday physical mechanisms comes much closer to solving the problem of specifying applicability. In each of the rubber band examples, various pieces of intuitive physical knowledge describe the mechanism at work: the rubber band binds the newspaper, grips the jar lid, and acts a source of springiness for the bobbing object. Although a mapping cannot be made from the rubber band to scientific entities, it is quite easy to map these qualitatively distinct physical processes to scientific entities and laws. For example, instances of binding almost always map to a practically rigid body. Likewise, gripping maps to friction forces, and springiness maps to Hooke's law. This suggests that applicability can depend directly on our intuitive knowledge—knowledge that exists prior to any formal scientific training.

We have argued that there is often more similarity between expert and novice than meets the eye. Historically, elements of prior knowledge have played essential roles in the development of scientific theory. Prior knowledge has provided new concepts for scientific theory by abstracting objects and processes from everyday experience. In some cases of qualitative reasoning, intuitive conceptions like balancing stand in for more elaborate and precise theoretical formulations in problem solving. Prior knowledge of physical mechanisms provides the means to map the objects in problem situations to appropriate scientific concepts. Finally, prior knowledge supports problem formulation and simplification in situations that are difficult to reduce to first principles. Based on these examples, it seems more productive to study the roles that naive physical conceptions continue to play in expert reasoning than to suggest that the main issue in acquiring expertise is to remove and replace them.

TOWARD A CONSTRUCTIVIST THEORY OF LEARNING

In this final section, we identify a set of theoretical principles that represents a step beyond the epistemological premise of constructivism. These principles serve as a framework for reinterpreting and reevaluating the results of misconceptions research and of orienting future empirical studies. Most important, they provide multiple ways for explaining how novice conceptions, including common misconceptions, play productive roles in acquiring more advanced mathematical and scientific understandings. They also begin to provide more detailed theoretical descriptions of knowledge and learning processes than are found in many accounts of learning. We begin with four basic principles about the nature of knowledge and learning in mathematics and science.

Fundamental Commitments

Knowledge in pieces: An alternative to "F = ma in the mind."
Presumptions about the diversity and grain size of knowledge involved in mathematical and scientific expertise have typically been too few and too large. Traditional analyses of expert reasoning have focused the use of powerful, general pieces of core knowledge, such as $F = ma$ or conversion to common denominator. But the mathematical and scientific knowledge of both experts and novices is distributed across a far greater number of interrelated general and context-specific components than either those analyses of expertise or textbook presentations suggests. Evidence of the distributed nature of knowledge has been reported in various domains—addition of natural numbers (Siegler & Jenkins, 1989), order and equivalence of common fractions (Behr et al., 1984; Smith, 1990), and Newtonian mechanics (diSessa, 1983, 1988, 1993). Success in understand-

ing how expertise is achieved over time will depend in part on developing substantially more elaborate and detailed models of knowledge.

It is the nature of mathematical and scientific knowledge that the most elegant and valued expression consists of very general, compact, and abstract propositions. Formal descriptions of knowledge in mathematical and scientific disciplines have, in turn, exerted a strong influence over what knowledge we want students to learn and how we expect them to learn it. Textbook presentations rely on similar compact, general, and abstract propositions and procedures. The principle of knowledge in pieces expresses our conviction that these characterizations of disciplinary knowledge cannot provide adequate models of either novices' or experts' understandings if their real-time reasoning on nonroutine tasks is taken seriously and examined carefully.

In developing more adequate knowledge models, we should not limit the range of knowledge components we try out. Mathematical knowledge has been analyzed in terms of the strategies students use to solve tasks representative of a conceptual domain (Behr et al., 1984; Siegler & Jenkins, 1989; Smith, 1990). In Newtonian mechanics, diSessa (1983, 1993) characterized prior intuitive knowledge in terms of *p-prims*—minimal explanatory abstractions of experiences in the day-to-day physical world. Many of these are expressed as qualitative proportionalities: "the more X, the more Y," as in "more effort leads to greater results" (diSessa, 1993; Roschelle, 1991). There is growing evidence that experts also use diverse types of knowledge to find and appropriately apply more traditional, disciplinary forms of knowledge like physics principles and mathematical theorems (diSessa, 1993; Rissland, 1985). Mental constructs that explain how computer programmers construct and understand their programs range from *plans* and *templates* (Linn, Katz, Clancy, & Recker, 1992; Spohrer, Soloway, & Pope, 1989) to mental models (diSessa, 1991; Young, 1983). As our last case analysis illustrated, experts' application of physical laws can depend on their knowledge of physical mechanisms. Both experts and novices have intuitions, abstract as well as concrete knowledge, and general as well as specific knowledge.

Besides issues of number and grain size, knowledge in pieces brings some different dimensions to the analysis of knowledge than are typically presumed in disciplinary models. Both misconceptions and disciplinary models suggest that knowledge is unitary, stable, and static, whereas richness and generativity are central properties of both expert and novice reasoning. Richness can be seen in novices' diverse ways of viewing and describing the world, and generativity in the inventiveness of their explanations. Similarly, experts are not just skilled and automatic performers. They constantly reconstruct the logic and breadth of their field and adapt flexibly to circumstances they have not encountered before. They sometimes make mistakes, but they can correct those mistakes fluently. A shift toward viewing knowledge as involving numerous elements of different types seems crucial to capture these features.

Continuity

Constructivism demands that more advanced states of knowledge are psychologically and epistemologically continuous with prior states. The principle of knowledge in pieces provides different ways to conceptualize this continuity. If expert knowledge is not adequately modeled by a few general, abstract principles like $F = ma$, there are many more opportunities for novice conceptions to play direct roles in expert reasoning. If students' ideas are productive in some contexts, and expert knowledge is more diverse than disciplinary models suggest, then the former can play more substantive and positive roles in learning.

It is impossible to separate students' misconceptions, one by one, from the novice knowledge involved in expert reasoning. Efforts to distinguish valid from invalid conceptions (e.g., preconceptions as distinct from misconceptions; Glaser & Bassok, 1989) are suspect when they fail to fairly assess the range of application of those ideas. Persistent misconceptions, if studied in an evenhanded way, can be seen as novices' efforts to extend their existing useful conceptions to instructional contexts in which they turn out to be inadequate. Productive or unproductive is a more appropriate criterion than right or wrong, and final assessments of particular conceptions will depend on the contexts in which we evaluate their usefulness. Teachers and researchers cannot overlook the power they exercise in choosing the situations and tasks in which students' knowledge is assessed. Judging the productiveness of students' conceptions demands a broad view of applicability.

Continuity opens new ways to conceptualize the evolution of expert understanding. The replacement of misconceptions gives way to knowledge refinement as a general description of conceptual change. Old ideas can combine (and recombine) in diverse ways with other old ideas and new ideas learned from instruction. Theories of learning that emphasize the refinement of prior conceptions must be informed by deep analyses of mathematical and scientific expertise—that is, they must know where learning is going, but they must also show in appropriate detail how expertise is acquired from the resources initially provided by more naive states. An adequate theory of learning must both provide richer descriptions of knowledge and explain the gradual transformation of that knowledge into more advanced states.

Functionality

Functionality, like continuity, is epistemologically fundamental to constructivism. Learning is a process of finding ideas that sensibly and consistently explain some problematic aspect of the learner's world. Conceptions that do not work in this way (or are linked to other conceptions that do) are unlikely to take root, be applied in reasoning, and subsequently defended by

students. Functionality, in turn, demands some account of success itself. Different sources—internal, social, and physical—must generate feedback on repeated efforts to act, control, and understand, and this feedback must be evaluated. Functionality implies that misconceptions, at least those reported to be widespread, carry with them some contexts of successful use. Taking this principle seriously provides an excellent heuristic to discover the origins of novice conceptions.[13] The constructivist position that one learns from trying what one currently knows casts functionality as underlying and framing continuity. But functionality is also constrained by continuity: Judgments of the success of any conception depend on the learner's existing knowledge and criteria of sense making.

A Systems Perspective

In emphasizing the continuity principle over the simple distinctions of correct expert knowledge versus misconceptions, abstract versus concrete knowledge, and general versus specific knowledge, it may seem we have sidestepped the task of understanding expertise. This is not so. Instead, we argue for an analytical shift from single units of knowledge to systems of knowledge with numerous elements and complex substructure that may gradually change, in bits and pieces and in different ways. There are indeed substantive differences between novice and expert knowledge, but they cannot be assessed one element at a time. Our focus must expand to the level of systems. The central task of a constructivist theory of learning is to establish, at a fine grain of detail, how novice knowledge systems evolve into expert ones.

The shift from particular conceptions to complex knowledge systems substantially changes how we evaluate individual conceptions. If they are embedded in complex systems, it is much easier to understand how some conceptions can fail in some contexts and still play productive roles overall. Some conceptions may come to play small but necessary roles in expert reasoning; others will become irrelevant without being replaced. Contexts of application may shift rather than the conceptions themselves, and even the descriptive vocabulary that defines contexts may change substantially. Overall, learning a domain of elementary mathematics or science may entail changes of massive scope. New elements may gradually come to play central roles as core knowledge, creating very large ripple effects through the system. System-level differences in overall stability and in the kinds and strength of relationships (or fragmentation) between elements are also possible.

A knowledge system perspective does not limit the form of constituent elements; it can support diverse knowledge types, like justification, strategy, and control knowledge, as well as more traditional categories of concepts and principles. This

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[13]diSessa (1993) discusses this and a range of other heuristics for discovering and developing adequate accounts of naive knowledge.

flexibility permits innovation in characterizing the diversity and complexity of cognition. New types of knowledge elements and subsystems may be necessary, such as mental models (Gentner & Stevens, 1983), *aesthetics* (diSessa, 1993), p-prims (diSessa, 1983), registration and qualitative cases (Roschelle, 1991), and domain-specific beliefs about the nature of knowledge and learning (Hammer, 1991; Schoenfeld, 1983, 1985). Characterizing how these elements and subsystems, once identified, interact to produce the real-time reasoning and problem solving of experts and novices is a major theoretical task.

Methodological and Pedagogical Issues

Our central theoretical assertion—knowledge viewed as a complex system of numerous elements—has important methodological and pedagogical implications for future research. We highlight some of the most central very briefly.

Task design.   The principle of knowledge in pieces has direct implications for the design of assessment tasks. It increases the demands on such tasks to fairly represent the range of knowledge and reasoning in specific conceptual domains. Different tasks—each apparently assessing the same expert concept—will be required if we expect to discover the full range of relevant novice knowledge and characterize its properties. Students who view physical situations in simulations may make different predictions than when viewing static presentations (Kaiser, Proffitt, & Anderson, 1985) or think that the laws of motion in a vacuum are very different than those on earth (Minstrell, 1989). Changing our expectations about the character of novice (and expert) knowledge from a few general principles to many interrelated and context-specific components requires a corresponding revision of our assessments methods.

Assessment must be sensitive to the breadth and generality of the novice knowledge system in its own terms. We must look for the competence and the potentially general comprehension strategies of novices as much as for their evident incompetencies and sensitivity to context. Simple, qualitative tasks, appropriately designed, have been useful in this regard, and keeping an eye open for unexpected knowledge is a central methodological heuristic. Similarly, assessing experts with tasks that are trivial and overlearned for them does a disservice to the justification structure, stability, flexibility, and generativity of their knowledge. It may be more illuminating to see how rusty experts rebuild their competence than to watch experts simply exercise it.

Inferring knowledge elements.   Knowledge elements—the basic units of analysis in the complex systems perspective—are identified by

carefully examining subjects' performance in solving nonroutine tasks that tap some domain-specific competence. The identification of knowledge elements must involve some abstraction from specific responses and from the particular features of the task situation. DiSessa (1993) has presented a set of methodological heuristics for identifying and characterizing knowledge elements and their interrelations. More generally, future research must develop a terminology that describes the theoretical character of knowledge elements (i.e., their form and function in the larger system) without assuming that the familiar categories of disciplinary knowledge (theorems, principles, etc.) will be sufficient. Theoretical invention and clarity will, in turn, motivate more sophisticated techniques of collecting and analyzing data.

Targeting knowledge systems.   Future research in mathematics and science learning should deliberately select, analyze, and empirically evaluate knowledge systems in particular domains. Some progress is possible by characterizing knowledge elements without specifying the properties of the system in which they are embedded, but other questions about knowledge elements can be appropriately addressed only within some system-level framework. Specifically, progress in characterizing application context, relations between elements, and processes of gradual change do not seem possible without making and testing claims about the nature of the system as a whole.

Discussion rather than confrontation.   Classroom discussion, when freed of its confrontation frame, can play an important role in learning, particularly when it concerns problematic situations in which students' ideas are strongly engaged and the impact of reformulation may be most clear. But the purpose of discussion changes when we conceptualize learning in terms of refinement rather than replacement. We still need to have students' knowledge—much of which may be inarticulate and therefore invisible to them—accessed, articulated, and considered. Rather than opposing those ideas to the relevant expert view, instruction should help students reflect on their present commitments, find new productive contexts for existing knowledge, and refine parts of their knowledge for specific scientific and mathematical purposes. The instructional goal is to provide a classroom context that is maximally supportive of the processes of knowledge refinement.

Analytic microworlds.   Constructivists have long emphasized physical materials that foster interactive learning and reflection (Resnick & Ford, 1981). New computer-based efforts continue this tradition with more flexible and interactive media. In addition to their traditional role in supporting learning, computer-based microworlds also offer special opportunities to

expose the content and form of students' knowledge and learning patterns to researchers. The nature and purpose of these microworlds are diverse. Some, like simulation environments and computer-based graphing packages, extend what students are able to do with traditional media (Pea, 1985). Others, like the *Envisioning Machine,* have been designed to generate and support reasoning about particular tasks in specific conceptual domains (Roschelle, 1992). A third class of microworlds can support students' efforts to map out the organization and ontology of their existing understandings in *knowledge spaces* (diSessa, 1990).

CENTRAL ASSERTIONS REVISITED

We conclude by reviewing the core assertions of misconceptions research and offering a revised formulation of each one. This reinterpretation is again far from a rejection of misconceptions research; it accepts—indeed is built on—the validity of the empirical findings of that research. The fact that students have mathematical and scientific conceptions that are faulty in a variety of contexts can be reframed to highlight their useful and productive nature as well as their limitations. This reconceptualization is essential for future progress in constructivist research in mathematics and science learning.

Casting Misconceptions as Mistakes Is too Narrow a View of Their Role in Learning

Misconceptions research has proven beyond any doubt that students' conceptions of mathematical and scientific phenomena fall short, or fail altogether, in many tasks and situations, when judged by the norms of the mathematical and scientific disciplines. Yet, assessments of students' ideas that focus only on their mistaken qualities are equally flawed. Constructivism asserts that prior knowledge is the primary resource for acquiring new knowledge, but misconceptions research has failed to provide any account of productive prior ideas for learning expert concepts and has overemphasized the discontinuity between novice and expert. With only unproductive misconceptions as potential resources for learning, achieving more sophisticated mathematical or scientific understandings is impossible.

In contrast to the misconceptions focus on discontinuity, there is substantial evidence that the form and content of novice and expert knowledge share many common features. Comparisons of experts and novices on the traditional dimensions of specific versus general knowledge, concrete versus abstract knowledge, or intuitive versus formal knowledge have not been evenhanded. Novice knowledge systems, as well as more

expert ones, can contain both general and abstract elements and specific and concrete components (diSessa, 1993). Expert reasoning centrally involves prior, intuitive knowledge that has been reused or refined. These shared features provide continuity between prior and more advanced states and the basis for conceptualizing the complex and subtle changes involved in learning even elementary mathematics and science. Further progress in understanding knowledge and learning will require more theoretically adequate accounts of conceptual change than misconceptions provides. We believe the central move is to start understanding knowledge as a complex system.

Misconceptions Are Faulty Extensions of Productive Prior Knowledge

We agree that the origins of misconceptions lie in prior experience and learning, inside and outside of classrooms. But the search for the origins of those misconceptions is not a matter of locating the root of an educational problem. Conceptions that lead to erroneous conclusions in one context can be quite useful in others. "Motion implies a force," although inadequate in many mechanical situations, provides a reasonable explanation of why electrical current flows in proportion to voltage. "Multiplication makes numbers larger" is an accurate general characterization of the effect of multiplication on natural numbers—a very large and frequently used, if restricted set of numbers. That this conception fails to adequately characterize multiplication with rational and real numbers does not relegate it simply to the status of a mistake. Most, if not all, commonly reported misconceptions represent knowledge that is functional but has been extended beyond its productive range of application. Misconceptions that are persistent and resistant to change are likely to have especially broad and strong experiential foundations.

Misconceptions Are Not Always Resistant to Change; Strength Is a Property of Knowledge Systems

Misconceptions can be found in most domains of mathematics and science, but not all are stable and resistant to change. Appropriately designed interventions can result in rapid and deep conceptual change in relatively short periods (Brown & Clement, 1989). Some misconceptions may persist simply for lack of plausible alternatives (again, see Brown & Clement, 1989); others, because they are part of conceptual systems that contain many useful elements whose breadth and utility are not immediately apparent. Understanding the strength of a particular conception will depend on a characterization of the knowledge system that embeds that element.

Interference is a biased assessment of the role of novice conceptions in learning. Though they may be flawed and limited in their applicability, novice conceptions are also refined and reused in expert reasoning.

Learning requires the engagement and transformation of productive prior resources, and misconceptions, when taken as mistakes, cannot play that role. Alongside conceptions that appear to interfere with learning are other ideas that can be productively engaged and developed (e.g., Minstrell & diSessa, 1993). Given appropriate instruction, those conceptions can serve as anchors in the process of building a more expert-like understanding (Clement et al., 1989). It is, however, practically impossible to categorically separate novice conceptions that are fundamentally flawed (misconceptions) from those that support learning expert concepts. Assessments of the worth of novice conceptions must be indexed to specific contexts of application because simple shifts in application context can turn wrong answers into productive ideas. Learning difficult mathematical and scientific concepts will never be effortless, but neither will it be possible at all without the support, reuse, and refinement of prior knowledge.

Replacing Misconceptions Is Neither Plausible Nor Always Desirable

Replacement—the simple addition of new expert knowledge and the deletion of faulty misconceptions—oversimplifies the changes involved in learning complex subject matter. By remaining mute on the processes and the specific conceptual resources involved in learning, replacement is similar to *tabula rasa* models of learning in asserting that any new acquisition is possible. Literal replacement itself cannot be a central cognitive mechanism (Smith, 1992), nor does it even seem helpful as a guiding metaphor (Bloom, 1992). Evidence that knowledge is reused in new contexts—that knowledge is often refined into more productive forms—and that misconceptions thought to be extinguished often reappear (e.g., Schoenfeld et al., 1993) all suggest that learning processes are much more complex than replacement suggests. Appreciating the broader applicability of some misconceptions may make even the goal of replacement less attractive (Smith, 1992). To avoid defaulting to replacement models, researchers should begin to formulate alternative learning mechanisms that can account for the complexity of students' ideas and undertake research to evaluate those models.

Instruction That Confronts Misconceptions Is Misguided and Unlikely to Succeed

Instruction designed to confront students' misconceptions head-on (e.g., Champagne et al., 1985) is not the most promising pedagogy. It denies the validity of

students' conceptions in all contexts; it presumes that replacement is an adequate model of learning; and it seems destined to undercut students' confidence in their own sense-making abilities. Rather than engaging students in a process of examining and refining their conceptions, confrontation will be more likely to drive them underground. But questioning the instructional effectiveness of confrontation does not imply that novice conceptions are valid in all contexts, only that their usefulness in some contexts must be respected. Targeting particular misconceptions for confrontation and replacement overemphasizes their individual importance relative to broader system-level issues. The goal of instruction should be not to exchange misconceptions for expert concepts but to provide the experiential basis for complex and gradual processes of conceptual change. Cognitive conflict is a state that leads not to the choice of an expert concept over an existing novice conception but to a more complex pattern of system-level changes that collectively engage many related knowledge elements.

It Is Time to Move Beyond the Identification of Misconceptions

Now that misconceptions are recognized as a pervasive phenomenon in mathematics and science learning, research that simply documents them in yet another conceptual domain does not advance our understanding of learning. We now need research that focuses on the evolution of expert understandings in specific conceptual domains and builds on and explains the existing empirical record of students' conceptions. Especially needed are detailed descriptions of the evolution of knowledge systems over much longer durations than has been typical of recent detailed studies (e.g., Confrey, 1988; Schoenfeld et al., 1993). This work should include not only rich case studies, but also explicit theoretical frameworks of knowledge systems and studies that evaluate the generality of those models.

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APPENDIX A

Dimensions of Variability in Misconceptions Terms

There are at least two dimensions of variability among the terms used to denote students' flawed mathematical and scientific conceptions. The adjective or prefix adjoined to conception or belief indicates variation along an epistemological dimension. The prefix to the most common term—misconception—emphasizes the mistaken quality of student ideas. Terms that include the qualifier—alternative—indicate a more relativist epistemological perspective. Students' prior ideas are not always criticized as mistaken notions that need repair or replacement but are understood as understandings that are simply different from the views of experts. The history of theory change in the natural sciences is often cited in this interpretation of student ideas and their evolution (Kuhn, 1964; Wiser, 1989). Students' alternative conceptions are incommensurable with expert concepts in a manner parallel to scientific theories from different historical periods. A third group of terms occupies an intermediate position along the epistemological dimension. Preconceptions and naive beliefs emphasize the existence of student ideas prior to instruction without any clear indication of their validity or usefulness in learning expert concepts. However, researchers who have used them have tended to emphasize their negative aspects.

This epistemological dimension emphasizes differences in content. The content of student conceptions—whether mistaken, preexisting, or alterna-

tive—are judged in contrast to the content of expert concepts. But these terms also fall out along a second dimension that distinguishes them on the basis of their organization. The terms conceptions and beliefs focus on individual ideas and do not directly implicate any larger scale cognitive structure that contains them. In contrast, framework and theory suggest that particular student conceptions are embedded in larger scale structures that integrate and interrelate those ideas. Thus, the psychological claims made by researchers with this latter theoretical perspective are stronger than those that focus on individual conceptions or beliefs. They must show not only that students have the particular conceptions but also that those ideas are systematically related in a theory and framework structure.

Despite these important variations, all terms assert differences between the ideas that students bring to instruction and the concepts presented in the classroom. We see these terms and distinctions as part of a broader pattern of knowledge terms that establish and maintain the split between novice understanding and more expert knowledge. It is common, for example, to contrast the formal knowledge of instruction with the intuitive or informal knowledge that students bring to the classroom (Hiebert & Behr, 1988; Resnick, 1986; Shaughnessy, 1977, 1985). One effect of employing this terminology—whether intentional or not—is to attribute lower status to informal knowledge.[14] What students learn from instruction is considered formal knowledge. It is highly valued and attributed the stability and systematicity of formal theoretical systems, in which instructed knowledge might be a more apt term. Part of the difficulty with the distinction derives from the various meanings of formal. One sense of formal knowledge derives from formal schooling. Knowledge can be formal because the instruction that presents it is highly organized within the equally organized practice of formal schooling. Another sense of formal knowledge, however, derives from formal methods, which in the case of mathematics and science means logical, deductive methods. Likewise, Piaget's notion of formal operations was the model for thought that was both mathematically formal and epistemologically more advanced than previous cognitive states. In contrast, informal knowledge is generally valued only as a steppingstone to the formal. It may play a role in learning formal knowledge, but its ultimate role in expert reasoning is usually considered insignificant. In assigning differential status to formal and informal knowledge, this distinction asserts a fundamental difference between student-generated ideas and those that they acquire from instruction.

A similar distinction is the contrast between experts and novices in various mathematics and scientific domains (Glaser & Chi, 1988). Research

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[14]We acknowledge those researchers who have valued students' informal knowledge in their studies of student learning (see Voss, Perkins, & Segal, 1991). The existence of that work, however, does not contradict the assertion of a status differential between what is deemed informal and formal.

conducted within this paradigm has analyzed the problem solving and reasoning of established experts in various fields—physics most extensively—and contrasted it with that of beginning students. The results of these analyses emphasized fundamental differences between the two groups. Typically, novices were shown to focus on the surface features of the objects in the problems, whereas experts were able to quickly penetrate to the deep structural features that support correct and efficient solutions. If there were similarities in the reasoning of experts and novices, they were rarely reported. The main impact of the expert–novice distinction was similar to the formal–informal distinction; it emphasized the fundamental differences between students and experts and deemphasized any potential similarities.

APPENDIX B

Replacing Misconceptions as a Model of Learning

We have claimed misconceptions researchers have frequently understood learning mathematics and science as a process of removing (or unlearning) misconceptions and adding relevant expert concepts. Because the claim that replacement is a central assertion of misconceptions research is more interpretive than the other assertions—and perhaps more controversial—we offer several lines of argument to support our claim.

Replacement Dominates Explicit Descriptions of the Learning Process

Writing about instruction in Newtonian mechanics, McCloskey (1983) explicitly invoked replacement: Students give up their old views for new ones. He suggested that the success of replacement depends on how well instruction makes the advantages of the expert view clear to students.

> [Physics instructors] should discuss with their students their naive beliefs, carefully pointing out what is wrong with these beliefs, and how they differ from the views of classical physics. In this way students may be induced to give up the impetus theory and accept the Newtonian perspective. (p. 319)

G. J. Posner and colleagues, whose position has been influential among misconceptions and conceptual change researchers, also used replacement to describe the learning process. In early work, Posner and Gertzog (1982) suggested, "The following example from Petrie [1976] captures the essence of the process [of conceptual change] as we understand it at this time."

> The process of *giving up* [italics added] the concept of impetus and *replacing* [italics added] it with the concept of Newtonian motion in a straight line unless

acted upon by external forces is an example of what I mean by conceptual change. (p. 205)

Posner et al. (1982) explicitly developed their theory in terms of replacement.

> We thus express our theory of accommodation in response to two questions: (1) under what conditions does some central concept come to be *replaced* [italics added] by another?, (2) what are the features of a conceptual ecology which governs the selection of new concepts? (p. 213)

The central features of effective replacement, according to these authors, are rational and involve choosing a new concept to displace an old one. "Generally, a new conception is unlikely to displace an old one, unless the old one encounters difficulties, and a new intelligible and initially plausible conception is available that resolves these difficulties" (p. 220).

More recently, Strike and Posner (1985) presented a broader view that embraces both replacement and assimilation to existing cognitive structures:

> The important questions are the way learners incorporate new conceptions into current cognitive structures and the way they *replace* [italics added] conceptions which have become dysfunctional with new ones. (p. 212)

Alternative Learning Processes Are Either Not Articulated or Not Emphasized

If replacement is not the only kind of conceptual change, alternative processes must be identified. Along with Strike and Posner (1985), many researchers have proposed learning processes that involve new knowledge fitting in with, rather than replacing, existing conceptions. For example, learning has been described as reorganization or integration of prior beliefs with scientific ones. Yet, the nature of reorganization, restructuring, and integration have not been spelled out in any detail, and descriptions of learning are still dominated by the replacement model.

Champagne et al. (1985) emphasized replacement in describing their approach to teaching mechanics, while also referring to integration.

> Certain concepts and propositions in the uninstructed students' schemata *must be replaced* [italics added] by or integrated with the concepts and propositions of the physics expert's schemata. (p. 77)

The rest of the paper, however, primarily described the process of *ideational confrontation,* in which old mistaken ideas are confronted by new expert ones. Similarly, Snow (1989) characterized subject-matter learning as a two-step process that results in the replacement of misconceptions. While mentioning restructuring, he emphasized unlearning misconceptions.

[Preconceptions, alternative views, and naive theories] can all be serious impediments to learning if instruction does not detect and deal with them. The view of learning as restructuring and *replacing* [italics added] old beliefs implies that transition involves unlearning as much as it does learning. (p. 9)

Replacement Is Consistent With Other Assertions About Misconceptions

From the assertion of interference, it is a short step to the presumption that old conceptions must be removed to neutralize their negative effects. If misconceptions are not replaced, they will continue to interfere. On the other hand, if learning expert concepts involves an integration with misconceptions, then the generality of interference of misconceptions is questioned.

Two other central terms in the misconceptions literature, *overcoming* and *confrontation,* are also consistent with the replacement model. If misconceptions must be overcome, they are suppressed, presumably in favor of more adequate conceptions. This is tantamount to replacement. With respect to confrontation, we recall the threefold process leading to replacement that was described by McCloskey: (a) building instructed alternatives, (b) confronting existing misconceptions, and (c) accepting new ideas and rejecting old ones. Although unusually bold and unqualified in its statement, McCloskey's position has not been unusual in its essence.